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UNLOADING WAVES FOR COMBINED LONGITUDINAL AND TORSIONAL PLASTIC WAVES

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Unloading Waves for Combined

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Abstract

Assuming a one-dimensional rate independent theory of combined longitudinal and torsional plastic wave propagation in a thin-walled tube, it is shown that the velocity of unloading waves, c_u, must satisfy either $c_s < c_u < c_2$ or $c_f < c_u < c_o$ where c_s and c_f are respectively the velocities of slow and fast plastic waves of combined stress. c_2 and c_o are respectively the elastic shear wave speed and the elastic bar velocity. It is also shown that the velocity of loading waves (moving elastic-plastic boundaries across which loading takes place), c_ℓ, must satisfy $c_\ell < c_s$ or $c_2 < c_\ell < c_f$ or $c_o < c_\ell$. The general features of the discontinuities associated with each type of loading and unloading wave are established, and examples are presented of unloading waves overtaking simple waves.

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Introduction

Several investigators involved with one dimensional longitudinal plastic wave propagation have considered the problem of moving boundaries separating regions in which the material response is elastic from regions in which the material response is plastic. These moving elastic-plastic boundaries are called unloading waves if the material at a section changes from a plastic state to an elastic state as the wave passes the section. Similarly, a moving elastic-plastic boundary for which the material in front of the moving boundary is in an elastic state and behind is in a plastic state is called a loading wave. Lee [1][1] showed that on the basis of a strain-rate independent theory, the velocity, c_u, of an unloading wave across which stress, velocity, and strain are continuous must satisfy $c < c_u < c_o$ where c_o is the elastic bar velocity and c is the plastic wave speed for the stress state at the unloading wave. He also showed that the velocity of a loading wave, c_ℓ, must satisfy either $c_\ell < c$ or $c_o < c_\ell$. These results have proved to be very helpful in understanding the essential features of plastic wave propagation when unloading is involved, as in the case of pulse loading of a slender rod [2,3].

The previously mentioned investigations of unloading waves are for the case when only one stress component is non-zero. In this paper we consider unloading waves for the case of combined longitudinal and torsional plastic wave propagation in thin-walled tubes. For this case, if lateral inertia effects are neglected, there are two non-zero stress components; namely, the axial stress σ and the torsional shear stress τ. The governing equations, as well as solutions for step-loading cases, for an isotropic work-hardening material have been given in an earlier

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[1] Numbers in brackets refer to references listed at the end of this paper.

paper by the author [4]. In vector form the equations are

$$Aw_t + Bw_x = 0 \tag{1}$$

where

$$w = \begin{bmatrix} u \\ \sigma \\ v \\ \tau \end{bmatrix} \quad A = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & M & 0 & N \\ 0 & 0 & \rho & 0 \\ 0 & N & 0 & P \end{bmatrix} \quad B = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

The subscripts t and x in Eq. (1) denote respectively, partial differentiation with respect to time and with respect to distance along the tube axis; u and v are respectively the longitudinal and circumferential particle velocities; ρ is the material density. In elastic regions the coefficients M, N, P are constants

$$M = 1/E$$

$$N = 0$$

$$P = 1/\mu$$

where E is Young's modulus and μ is the modulus of rigidity. In plastic regions these coefficients depend on the stress state (σ,τ) and the stress-strain behavior of the material. Thus, in plastic regions,

$$M = 1/E + G(\sigma/\theta)^2$$

$$N = G\sigma\tau$$

$$P = 1/\mu + G\theta^2\tau^2$$

where G is the positive scalar function which appears in the equation relating the plastic strain rate to the normal to the yield surface f.

$$\dot{\varepsilon}_{ij}^{P} = G\frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{k\ell}} \dot{\sigma}_{k\ell} \tag{2}$$

For the case of isotropic work-hardening considered here, G is a function only of the yield stress k

$$k = [(\sigma/\theta)^2 + \tau^2]^{1/2} \qquad (3)$$

where, for the Tresca yield condition, $\theta = 2$, and for the von Mises yield condition, $\theta = \sqrt{3}$.

Unloading Waves

Consider an unloading wave $\phi(x,t) = 0$ as shown in Fig. 1 to be a wave across which stress and particle velocity are continuous. Let $w(x,t)$ be a solution of Eqs. (1) in both the elastic and the plastic regions. Thus,

$$A_e(w_t)_e + B(w_x)_e = 0; \text{ elastic region} \qquad (4a)$$

$$A_p(w_t)_p + B(w_x)_p = 0; \text{ plastic region} \qquad (4b)$$

where the subscripts e and p indicate that the associated quantity is evaluated in the elastic and plastic regions respectively. Since w is assumed to be continuous across the unloading wave, the total derivative of w along the wave front,

$$\frac{dw}{dt} = c_u w_x + w_t \qquad (5)$$

where $c_u = -\phi_t/\phi_x$ is the speed of the unloading wave, must also be continuous across the wave front. Thus, at a point P on the wave front

$$c_u(w_x)_e + (w_t)_e = c_u(w_x)_p + (w_t)_p \qquad (6)$$

Making use of Eqs. (4) we can write Eq. (6) as either

$$(c_uA_p - B)((w_t)_e - (w_t)_p) = c_u(A_p - A_e)(w_t)_e \qquad (7a)$$

or

$$(c_uA_e - B)((w_t)_e - (w_t)_p) = c_u(A_p - A_e)(w_t)_p \qquad (7b)$$

Eqs. (7) constitute a system of inhomogeneous linear equations for the jump in the time derivative of w, $[(w_t)_e - (w_t)_p]$. In contrast to the

problem of determining characteristic wave speeds, the unloading wave speed is not determined by the algebraic equations governing the jump in the time derivative of w across the wave front. On the other hand, if the unloading wave speed c_u is known, the jump in the time derivative of w can be computed from the solution on either side of the wave front provided that neither the determinant of $(c_u A_p - B)$ nor the determininat of $(c_u A_e - B)$ vanishes. The latter provision is simply the condition that the unloading wave speed not be equal to either of the two plastic wave speeds or either of the two elastic wave speeds.

Although Eqs. (7) do not determine the unloading wave speed they do place restrictions on permissible wave speeds for which unloading can occur. In order to exhibit these restrictions it is convenient to eliminate velocities from Eq. (7b) to obtain

$$(\rho c_u^2/E - 1)[(\sigma_t)_e - (\sigma_t)_p] = \rho c_u^2\, Gk(k_t)_p \sigma \qquad (8a)$$

$$(\rho c_u^2/\mu - 1)[(\tau_t)_e - (\tau_t)_p] = \rho c_u^2\, Gk(k_t)_p \theta^2 \tau \qquad (8b)$$

where $(k_t)_p$ denotes the time derivative of the expression in Eq. (3). In the plastic region $(k_t)_p$ is non-negative. The condition for unloading to take place at the wave front is the condition that $(k_t)_e$ be negative there. That is, for an unloading wave

$$(\sigma/\theta^2)(\sigma_t)_e + \tau(\tau_t)_e < 0 \qquad (9)$$

Substituting $(\sigma_t)_e$ and $(\tau_t)_e$ from Eqs. (8) in (9) and simplifying gives

$$\frac{L(\rho c_u^2)^2 - (M + P)_p(\rho c_u^2) + 1}{(\rho c_u^2/E - 1)(\rho c_u^2/\mu - 1)} < 0 \qquad (10)$$

where

$$L = \frac{1}{\mu E} + \frac{1}{\mu}G(\sigma/\theta)^2 + \frac{1}{E}G\theta^2\tau^2$$

The roots of the denominator of Eq. (10) are the elastic bar velocity $c_o = (E/\rho)^{1/2}$ and the elastic shear wave speed $c_2 = (\mu/\rho)^{1/2}$. Comparison of Eq. (10) with Eq. (18) of [4] reveals that the roots of the numerator of Eq. (10) are the slow and fast plastic wave speeds denoted by c_s and c_f in [4]. These plastic wave speeds satisfy the inequalities

$$0 < c_s \leq c_2 \tag{11a}$$

$$c_2 \leq c_f \leq c_o \tag{11b}$$

From inequalities (11), inequality (10) is satisfied only if c_u satisfies either

$$c_s \leq c_u \leq c_2 \tag{12a}$$

or

$$c_f \leq c_u \leq c_o \tag{12b}$$

Thus there are two, in general, distinct ranges of unloading wave speeds. Unloading waves with wave speeds satisfying (12a) will be referred to as slow unloading waves and those satisfying (12b) as fast unloading waves.

For a loading wave (k_t) must be positive on both sides of the wave front. This condition is equivalent to reversing the sign of the inequality in (10). Then, from inequalities (11) the speed of loading waves, c_ℓ must satisfy

$$c_\ell \leq c_s \tag{13a}$$

or

$$c_2 \leq c_\ell \leq c_f \tag{13b}$$

or

$$c_o \leq c_\ell \tag{13c}$$

Analogous results for loading waves have been obtained for a very general elastic-plastic continuum by Green [5].

Discontinuities at Unloading Waves

If the plastic region in front of the unloading wave is a constant state region, then $(k_t)_p$ is zero and, from Eqs. (8), the velocity of unloading waves must be equal to one of the elastic wave speeds and a discontinuity occurs in the time derivative of only one of the stresses. If the discontinuity is in τ_t then the unloading wave speed is equal to c_2 whereas if the discontinuity is in σ_t the unloading wave speed is equal to c_o.

If the plastic region in front of the unloading wave is not a constant state region, then jumps in both σ_t and τ_t generally occur. In order to understand the nature of the unloading behavior for the two types of unloading waves as well as the behavior for the three types of loading waves, it is helpful to investigate the directions in stress space of the jump in the time derivative of the stress vector associated with each of these waves. That is, we shall consider the jump in $\bar{\sigma}_t$ where $\bar{\sigma}$ is the vector with components σ and τ and the subscript t again denotes partial differentiation with respect to time. We shall refer to $[(\bar{\sigma}_t)_e - (\bar{\sigma}_t)_p]$ as the jump for unloading waves and $[(\bar{\sigma}_t)_p - (\bar{\sigma}_t)_e]$ as the jump for loading waves, where subscripts e and p again denote, respectively, evaluation on the elastic and plastic sides of the wave.

From Eqs. (8) the slope of the jump vector for unloading waves is

$$\frac{[\tau_t]}{[\sigma_t]} = \frac{(\rho c_u^2/E - 1)}{(\rho c_u^2/\mu - 1)} \frac{\tau}{(\sigma/\theta^2)} \qquad (14)$$

where [ ] denotes the jump in the enclosed quantity. Since c_u is always less than c_o, Eq. (8a) shows that $((\sigma_t)_e - (\sigma_t)_p)$ has the same sign as σ. Using the latter requirement to determine the sense of the jump vector and Eq. (14) to determine the slope, the range of directions for unloading from a stress state (σ,τ) for both fast and slow unloading waves is as shown in Fig. 2a. The jump vector is directed from the point (σ,τ) as origin. The limiting slopes from Eq. (14) as $c_u \to c_s$ and as $c_u \to c_f$ are respectively the slopes associated with slow and fast plastic acceleration waves. The latter directions, which are determined by the stress state (σ,τ) and the stress-strain behavior of the material, are indicated as $[\bar{\sigma}_t]_s$ and $[\bar{\sigma}_t]_f$ in Fig. 2. In [4] it is shown that $[\bar{\sigma}_t]_s$ and $[\bar{\sigma}_t]_f$ are mutually orthogonal and that the direction of $[\bar{\sigma}_t]_s$ lies between the direction of the positive τ-axis and the normal to the yield surface $\bar{n}$.

Eqs. (8) can be made applicable for the case of loading waves simply by replacing the unloading wave speed c_u by the loading wave speed c_ℓ. Then, the slope of the jump vector for loading waves is also given by Eq. (14) with c_u replaced by c_ℓ. The range of directions of the jumps for the three types of loading waves are shown in Fig. 2b.

Examples

The two types of unloading waves can be illustrated by considering unloading of simple wave regions. As a first example we consider the unloading situation in Fig. 3a where an unloading wave, resulting from a decrease in τ at the boundary, overtakes a slow simple wave. We shall show that the interaction results in a transmitted fast plastic wave, a transmitted slow unloading wave, a reflected elastic shear wave and a

reflected elastic longitudinal wave as shown in Fig. 3a. Here, as previously in the case of loading waves and unloading waves, a 'wave' is a curve in the t - x plane across which stress and particle velocity are continuous, but discontinuities occur in their derivatives (i.e. an "acceleration wave"). The regions between the various waves which intersect at P are numbered as regions 1 thru 6. The derivatives of stress and velocity at P can be discontinuous only across the six waves which intersect at P. Since the sum of the jumps taken along a closed curve surrounding P must be equal to zero we have

$$[w_t]_{2-1} + [w_t]_{3-2} - [w_t]_{3-4} - [w_t]_{4-5} = [w_t]_{6-1} + [w_t]_{5-6} \quad (15)$$

where, for example, $[w_t]_{2-1} = (w_t)_2 - (w_t)_1$. The right side of Eq. (15) is determined by the solution in the plastic region and the magnitude of the discontinuity in τ_t at the boundary. The left side can be written in terms of four unknown quantities. For this it is convenient to eliminate jumps in the time derivatives of the velocities by the relations

$$[u_t] = \frac{1}{\rho c} [\sigma_t] \quad (16a)$$

$$[v_t] = \frac{1}{\rho c} [\tau_t] \quad (16b)$$

where c is the wave velocity of the wave under consideration. Also, $[\sigma_t]_{2-1}$ and $[\sigma_t]_{6-1}$ can be eliminated by the relations

$$[\sigma_t]_{2-1} = \left(\frac{d\sigma}{d\tau}\right)_f [\tau_t]_{2-1} \quad (17a)$$

$$[\sigma_t]_{6-1} = \left(\frac{d\sigma}{d\tau}\right)_s [\tau_t]_{6-1} \quad (17b)$$

where $(d\sigma/d\tau)_f$ and $(d\sigma/d\tau)_s$ are slopes of the stress trajectories for fast

and slow simple waves respectively at the stress state corresponding to point P in Fig. 5a. Substituting for $[\sigma_t]_{3-2}$ amd $[\tau_t]_{3-2}$ from Eqs. (8) and using Eqs. (16) and (17), Eq. (15) can be written as

$$[\tau_t]_{2-1} + \frac{\rho c_u^2 \, Gk(k_t)_2 \theta^2 \tau}{(\rho c_u^2 /\mu - 1)} - [\tau_t]_{3-4} = [\tau_t]_{6-1} + [\tau_t]_{5-6} \qquad (18a)$$

$$\left(\frac{d\sigma}{d\tau}\right)_f [\tau_t]_{2-1} + \frac{\rho c_u^2 \, Gk(k_t)_2 \sigma}{(\rho c_u^2 /E - 1)} - [\sigma_t]_{4-5} = \left(\frac{d\sigma}{d\tau}\right)_s [\tau_t]_{6-1} \qquad (18b)$$

$$\frac{1}{\rho c_f} [\tau_t]_{2-1} + \frac{c_u Gk(k_t)_2 \theta^2 \tau}{(\rho c_u^2 /\mu - 1)} + \frac{1}{\rho c_2} [\tau_t]_{3-4} = \frac{1}{\rho c_s} [\tau_t]_{6-1} + \frac{1}{\rho c_2} [\tau_t]_{5-6} \qquad (18c)$$

$$\frac{1}{\rho c_f} \left(\frac{d\sigma}{d\tau}\right)_f [\tau_t]_{2-1} + \frac{c_u Gk(k_t)_2 \sigma}{(\rho c_u^2 /E - 1)} + \frac{1}{\rho c_o} [\sigma_t]_{4-5} = \frac{1}{\rho c_s} \left(\frac{d\sigma}{d\tau}\right)_s [\tau_t]_{6-1} \qquad (18d)$$

where

$$(k_t)_2 = (k_t)_1 + [k_t]_{2-1} = - [k_t]_{6-1} + [k_t]_{2-1}$$

Eqs. (18) constitute four equations in the four unknowns $[\tau_t]_{2-1}$, $[\tau_t]_{3-4}$ $[\sigma_t]_{4-5}$ and c_u. Eliminating the first three unknowns and simplifying we obtain the following equation for determining the unloading wave speed c_u

$$(QR - ST)[\tau_t]_{6-1} + 2Q[\tau_t]_{5-6} = 0 \qquad (19)$$

where

$$Q = - \frac{(\rho c_s^2 /E - 1)\theta^2 \tau (c_u/c_f - 1)(c_u/c_f + 1 + c_u/c_o + c_o/c_f)}{\sigma(\rho c_u^2 /E - 1)(\rho c_s^2 /\mu - 1)}$$

$$R = \frac{(c_u/c_s - 1)(c_u/c_s + 1 + c_u/c_2 + c_2/c_s)}{(\rho c_u^2/\mu - 1)}$$

$$S = \frac{(c_u/c_f - 1)(c_u/c_f + 1 + c_u/c_2 + c_2/c_f)}{(\rho c_u^2/\mu - 1)}$$

$$T = \frac{(\rho c_s^2/\mu - 1)\sigma(c_u/c_s - 1)(c_u/c_s + 1 + c_u/c_o + c_o/c_s)}{\theta^2\tau(\rho c_u^2/E - 1)(\rho c_s^2/E - 1)}$$

For a slow unloading wave, $c_s \leq c_u \leq c_2$, the coefficient Q is negative and the expression (QR - ST) is positive. The expression (QR - ST)/Q is a monotonically decreasing function of c_u which decreases from 0 at $c_u = c_s$ to $-\infty$ at $c_u = c_2$. Hence, the unloading wave speed c_u is uniquely determined for arbitrary negative values of $[\tau_t]_{6-1}$ and $[\tau_t]_{5-6}$. For strong unloading (i. e. large values of $[\tau_t]_{5-6}/[\tau_t]_{6-1}$) c_u approaches c_2 whereas for weak unloading c_u approaches c_s. Once c_u is determined from Eq. (19), the jumps $[\tau_t]_{2-1}$, $[\tau_t]_{3-4}$ and $[\sigma_t]_{4-5}$ can be determined from any three of Eqs. (18). The sign of $[\tau_t]_{2-1}$ is easily shown to be negative as required by Fig. 2. Thus, the general features of the interaction shown in Fig. 3a have been verified for arbitrary values of the unloading jump $[\tau_t]_{5-6}$, construction of the complete unloading boundary requires the use of numerical techniques.

If, at the boundary, the normal stress σ decreases while the shear stress τ remains constant, the resulting wave interaction is as shown in Fig. 3b. The analysis is the same as for Fig. 3a except that $[\tau_t]_{5-6}$ is now zero and the unloading is produced by the jump $[\sigma_t]_{5-6}$. The governing equation for the unloading wave speed c_u is the same as Eq. (19) with $2Q[\tau_t]_{5-6}$ replaced by $-2S[\sigma_t]_{5-6}$. Again c_u varies from c_s to c_2 as the ratio $[\sigma_t]_{5-6}/[\tau_t]_{6-1}$ varies from 0 to + ∞.

An example in which a fast unloading wave is generated is shown in Fig. 4. In this figure, unloading of a fast simple wave is shown to result in transmitted and reflected elastic shear waves, a reflected elastic longitudinal wave, and a transmitted fast unloading wave. The jumps across the waves must again satisfy Eq. (15). Proceeding as in the previous cases the following equation is obtained for determining the unloading wave speed c_u

$$Q[\tau_t]_{6-1} + 2[\sigma_t]_{5-6} = 0 \qquad (20)$$

where Q is the same as in Eq. (19). For $c_f < c_u < c_o$, Q is a positive, monotonically increasing function of c_u. From Eq. (20), c_u varies from c_f to c_o as the ratio $[\sigma_t]_{5-6}/[\tau_t]_{6-1}$, varies from 0 to $-\infty$.

Concluding Remarks

The results presented here were obtained under what would appear to be quite restrictive assumptions. Actually, the main conclusions of the analysis are valid under considerably less restrictive conditions. For example, the plastic wave speeds and the range of permissible wave speeds for both loading and unloading waves do not depend on the assumption of isotropic work-hardening, but depend only on the value of the scalar function G and the direction of the normal to the yield surface for the stress state at the wave. Also, the direction of the jump in the time derivative of the stress-vector across either a loading or an unloading wave does not depend on the assumption of isotropic work-hardening. The latter assumption was made in order to obtain explicit results that would provide insight into the essential features of un-

loading waves of combined stress.

The examples illustrate typical unloading wave behavior for unloading waves overtaking plastic waves. In the more general case where the plastic region in front of the unloading wave (region 1 in Figs. 3 and 4) is not a simple wave region and the boundary between regions 1 and 6 is itself an unloading wave, the qualitative features of the interaction are the same as shown in Figs. 3 and 4. That is, if the elastic wave of unloading overtakes an unloading wave with speed less than c_2 the reflected and transmitted waves will be as shown in Fig. 3 whereas if it overtakes an unloading wave with speed greater than c_2 the reflected and transmitted waves will be as shown in Fig. 4.

The case of an unloading wave meeting a simple wave has not been discussed. In this case the qualitative features of the wave interaction depend on the relative strengths of the waves tending to produce loading and unloading.

Finally, many qualitative features of the behavior along the unloading wave and in the elastic region (analogous to the results in References 2 and 3) could be obtained for the case shown in Fig. 4. However, it would be much more difficult, if not impossible, to obtain analogous results for the case shown in Fig. 3 because beyond point P the plastic region adjacent to the unloading wave is not a simple wave region.

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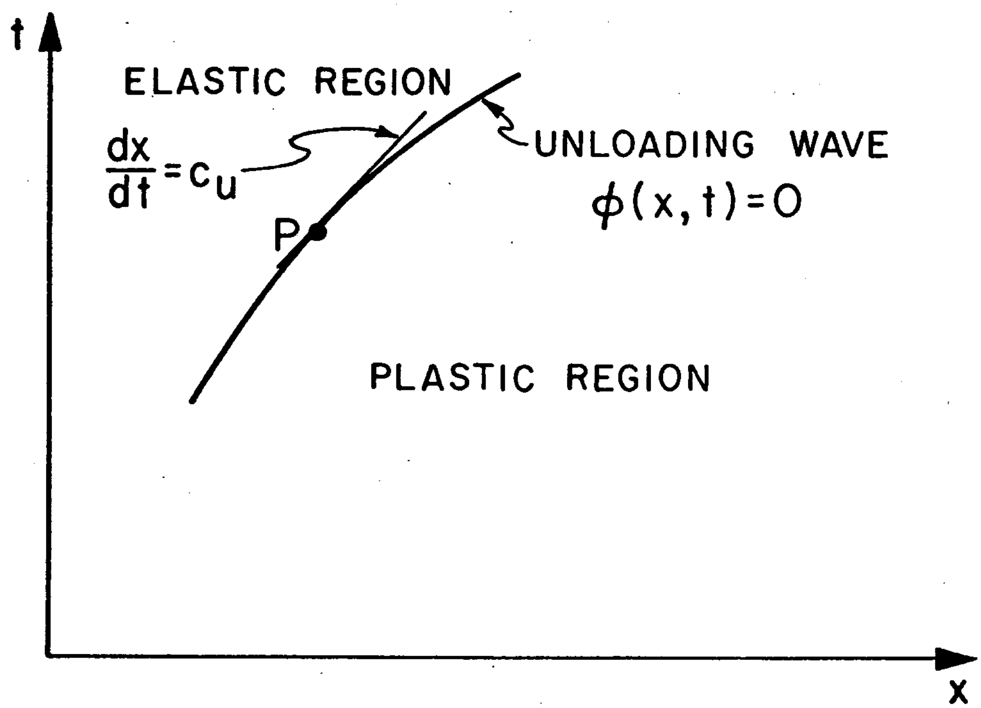

FIG. 1 AN UNLOADING WAVE

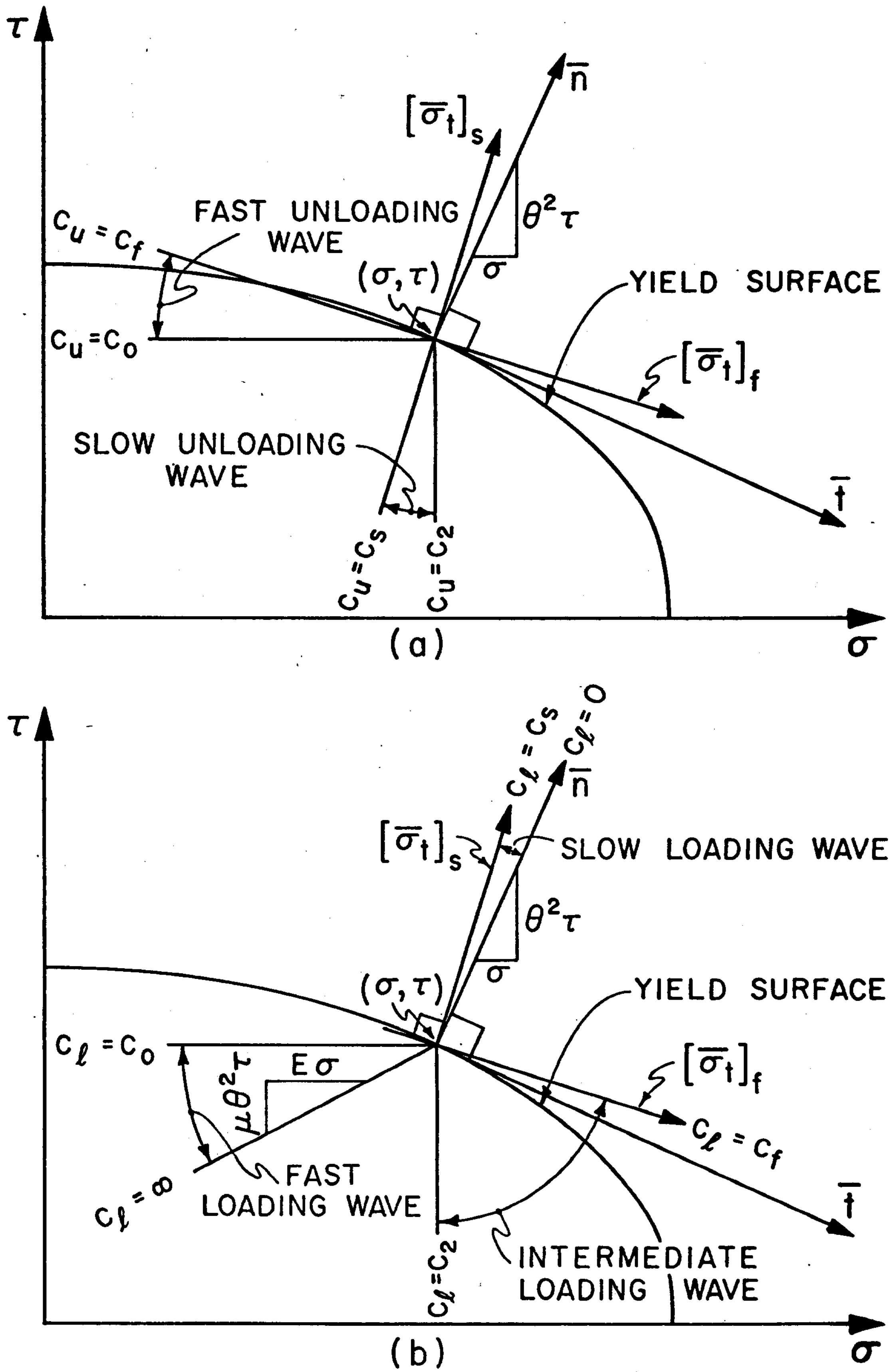

FIG. 2 RANGE OF POSSIBLE DIRECTIONS OF THE JUMP, $[\bar{\sigma}_t]$, ACROSS UNLOADING AND LOADING WAVES

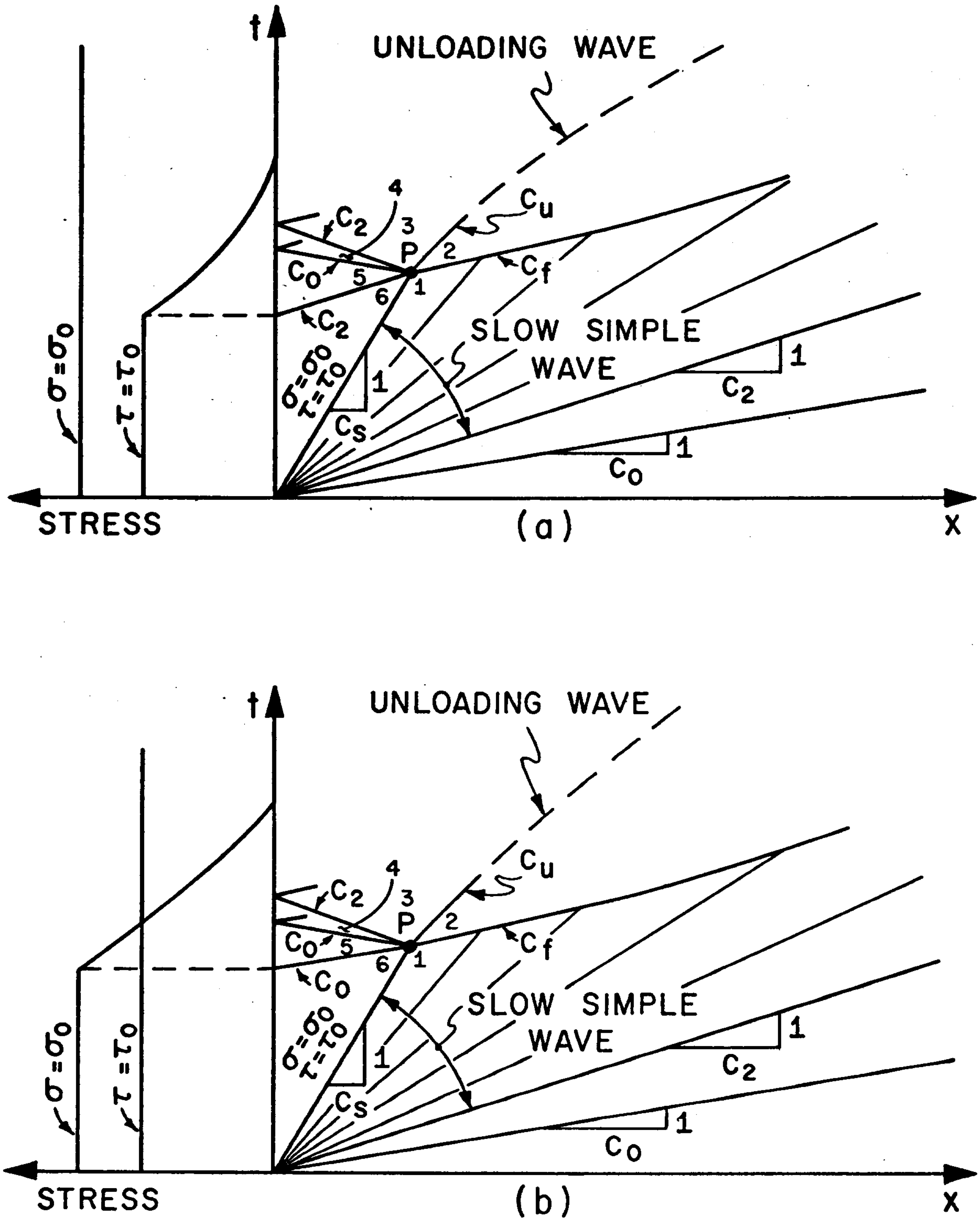

FIG. 3 UNLOADING OF A SLOW SIMPLE WAVE

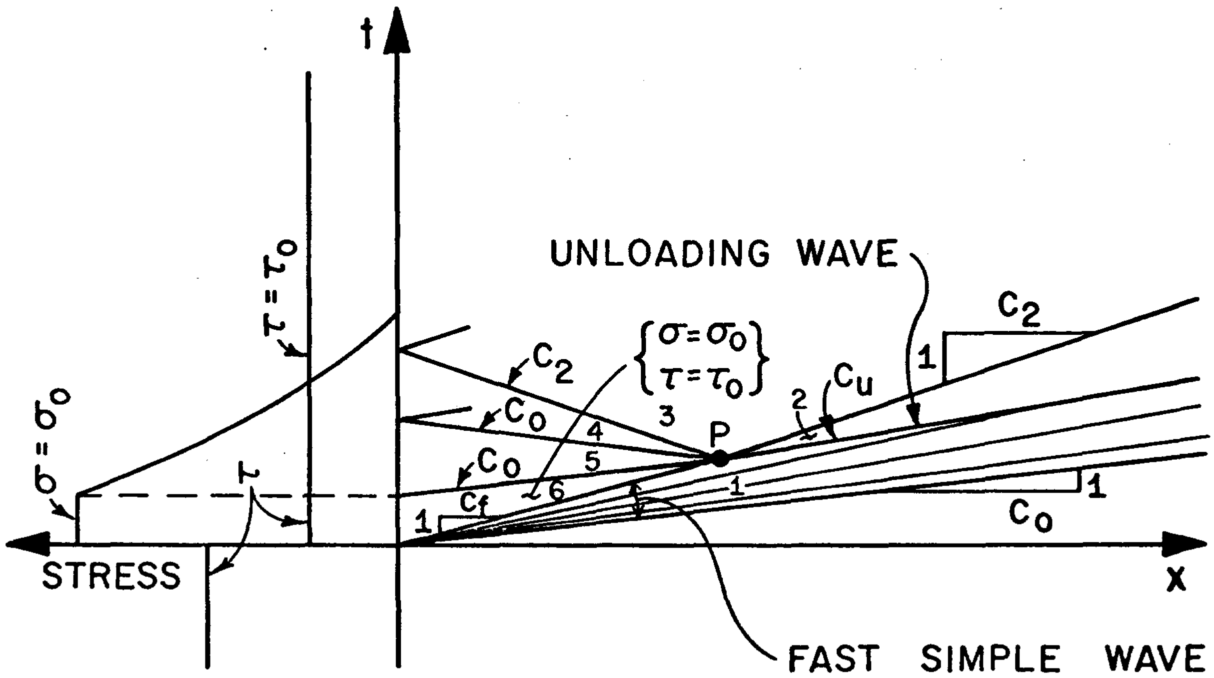

FIG. 4 UNLOADING OF A FAST SIMPLE WAVE